The tree in Pythagoras' garden

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This geometrical account of primitive Pythagorean triples was stimulated by a remark of Douglas Rogers on a recent paper by Roger Alperin (Alperin, 2005). Rogers, in commenting on this paper, noted that Fermat in the 17th century had posed a challenge problem on Pythagorean triples that suggested he knew how to construct a sequence of them, possibly via a geometrical method (Fermat, 1643). Rogers himself gave an expanded version of such a method and from this has come the present investigation.

Naturally, we are discussing Pythagoras' theorem, which states that:

> In any right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides. (These other sides, around the right angle, are sometimes called catheters, but the term 'legs' will be used for them in this paper.)

A standard Euclidean textbook proof of this theorem (see e.g., Forder, 1930) derives the result via congruence and area arguments applied to some triangles constructed from the basic figure. A non-standard proof, using similarity of certain ratios in a right-angled triangle, obtained via a perpendicular from the right-angled vertex to the hypotenuse, is simple, but algebraic instead of geometric. A much simpler direct argument, using only the additive properties of areas, is given by "shapes rearrangement" as shown in the Figure 1

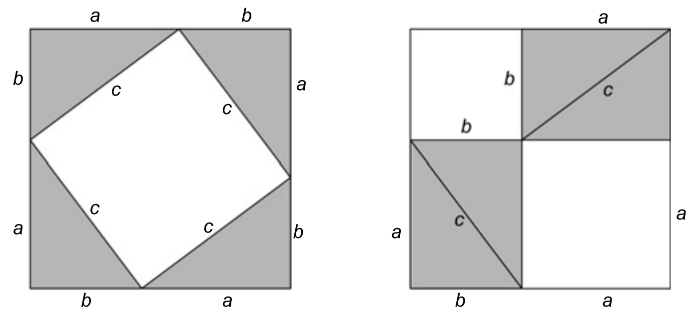

Figure 1. A direct argument of Pythagoras' theorem.

Our interest is in those cases where each side of the triangle has integer length, with the most famous example being given by the triple (3, 4, 5). Of course, if we multiply all the elements of such a triple by any positive integer, we obtain another valid triple.

There are other such triples that are not multiples of (3, 4, 5). These include for example (5, 12, 13), (8, 15, 17) or (20, 21, 29)—and which, like (3, 4, 5), the elements have no common factor. We shall call such triples (a, b, c) *Primitive Pythagorean Triples* (PPTs) if they satisfy the additional condition $a < b < c$.

There is a well-known method for constructing PPTs. It is based on the simple identity

$$(p - q)^2 + 4pq = (p + q)^2$$

Notice that two of the three terms in this identity are squares, and the other one is too, if we replace p and q by m^2 and n^2. Once we have done that we have the Pythagorean condition: a sum of two squares equalling a third square.

If we now make $m + n$ odd, choose $m > n$ and ensure they have no common factor, then the three sides $m^2 - n^2$, $2mn$, and $m^2 + n^2$ give

- the PPT $(m^2 - n^2, 2mn, m^2 + n^2)$ if $m < n$, and
- the PPT $(2mn, m^2 - n^2, m^2 + n^2)$ otherwise, i.e., if $m > n$.

(In fact, using a little number theory shows that the reverse is true—every PPT is describable in this way.)

Fermat's challenge

Fermat's 1643 challenge to his correspondents was to find six such PPTs in which the two legs differed in value by 1. We now describe a method for doing this.

It is well known that, associated with any triangle ABC, are its incircle and its three excircles. Figure 2 shows the case where the triangle ABC is right-angled at C. Note that all four circles are constructible by ruler and compass since their centres lie on the (internal or external) bisectors of the angles of ABC.

Using simple geometry it can be shown that, in the case of a right angled triangle ABC, the radii of these four circles are all determined by the side lengths a, b, c of the original triangle. In

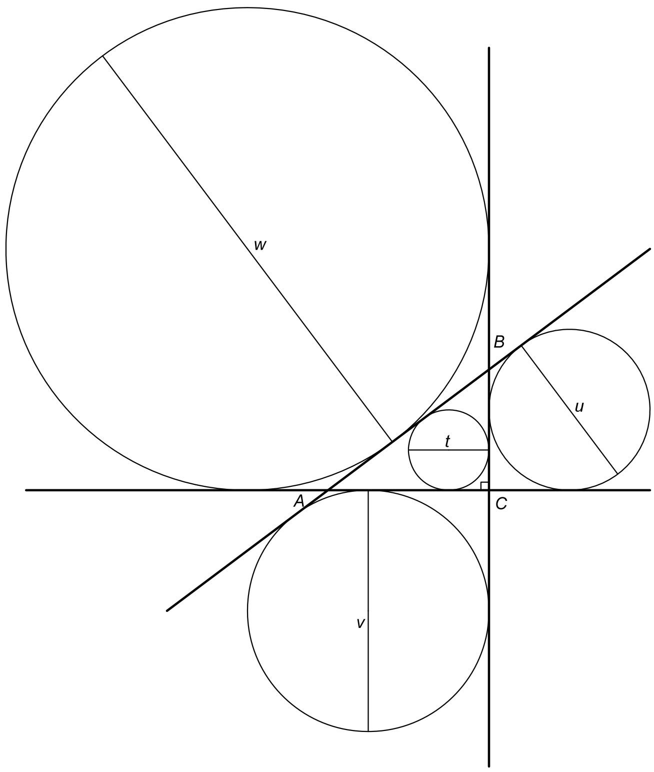

Figure 2. Diagram showing the incircle and three excircles for a right angled triangle. The right-angled triangle has its right angle at C. Sides are extended to allow easy views of the incircle and excircles (sometimes called i-circle and e-circles). The side opposite vertex A is designated a, etc. The diameter of the incircle is t and the diameters of the e-circles on sides a, b, c are u, v and w.

fact, denoting the diameter of the excircle on the hypotenuse by w, then:

$$w = a + b + c$$

and, since the RHS is always even for PPTs, it follows that the radius of this excircle is also an integer.

Corresponding calculations for the diameters u, v, of the excircles on the legs a, b, respectively show that:

$$u = a - b + c, \; v = -a + b + c$$

while the diameter, t, of the triangle's incircle is:

$$t = a + b - c$$

and, since each of these is even, all the corresponding radii are integers for any PPT.

Fermat's solution

Fermat's solution to his challenge problem is as follows:

The first such PPT is (3, 4, 5). Take double the sum of the three sides (i.e., $2w = 24$ in our notation), subtract in turn from it, each of the two legs, (so obtaining $24 - 3 = 21$ and $24 - 4 = 20$) and then add the hypotenuse to it (giving $24 + 5 = 29$).

Note that the difference between the two new legs is $21 - 20 = 1$, so that condition is preserved. Also $20^2 + 21^2 = 29^2$, so indeed we do have a second PPT satisfying his required condition.

All we have to do now is to repeat this a further 4 times to obtain a sequence of six PPTs in each of which the legs differ by 1, and we may clearly go on "forever". (In fact, the 6th PPT in the above sequence is, as Fermat gave, (23660, 23661, 33461).) A spreadsheet calculation is:

a	b	c
3	4	5
20	21	29
119	120	169
696	697	985
4059	4060	5741
23660	23661	33461

We do not know how Fermat obtained this method for solving his challenge problem. He was very good at finding relationships between quantities, and was also a prodigious calculator, so he may have found it by looking for a simple relationship connecting some PPTs that he knew satisfied his condition.

In general terms, we can write an algebraic expression which suggests the above method. For any x, a, and b, note that

$$(2x - a)^2 + (2x - b)^2 = 4x^2 + 4x(x - a - b) + a^2 + b^2 \qquad \text{[FF]}$$

If we now choose c so that (a, b, c) is a PPT, and choose $x = a + b + c$, the RHS becomes

$$4x^2 + 4xc + c^2 = (2x + c)^2$$

In addition, we have the result that, if (a, b, c) is a PPT, and if $w = x = a + b + c$ is the diameter of the excircle on the hypotenuse, c, then $(2w - b, 2w - a, 2w + c)$ is also a PPT. Moreover, the difference between the legs is preserved.

Fermat's second challenge

Fermat posed a second challenge, which was to find PPTs in which the difference between the legs remains at 7. Since (5, 12, 13) is such a PPT, all we have to do is to repeat the above argument but now starting with this PPT.

We now think of the above algebraic construction in purely geometrical terms.

- Starting with a PPT (a, b, c), construct the excircle on the hypotenuse, of diameter w as found above.
- "Blow up" this circle to one of diameter $2w + c$.
- Inside this new circle, using the diameter line that is parallel to the hypotenuse of the original triangle, construct the right-angled triangle with legs $2w - b$ and $2w - a$, as we know we can.
- Repeat this *ad infinitum* to obtain a chain of triangles all related by their hypotenuses.

Thinking of the situation geometrically as we have, we ask: "If this can be done using hypotenuses, can it also be done using each of the legs?"

In seeking an answer we go back to our fundamental formula [FF].

Suppose that we replace x by u, the diameter of the excircle on the shortest side, a.

We would then have, in the middle term, a factor $u - a - b = a - b + c - a - b$, which does not equal c—but it does equal c if we use $2x + b$ instead of $2x - b$ on the LHS of [FF]!

From this, we see that, using u instead of w will produce a sequence of PPTs of the form $(2u - a, 2u + b, 2u + c)$, all obtained by repeated use of the excircle on the shorter leg of the previous triangle. For example, starting with (3, 4, 5), we have $2u = 2(3 - 4 + 5) = 8$, so that $2u - a = 5$, $2u + b = 12$, $2u + c = 13$ and indeed (5, 12, 13) is a PPT.

Applying the same method to the longer leg b, where the excircle has diameter $v = b - a + c$, we find that the PPT constructed on the base triangle via the side b is $(2v - b, 2v + a, 2v + c)$. For example, starting with (3, 4, 5), for which $v = 6$, we obtain the PPT (8, 15, 17). Again, we may now construct an infinite sequence of PPTs using the longer leg in each successive triangle.

Summarising

What we have shown is that, given any right-angled triangle with sides forming a PPT, we can associate with each side a new right-angled triangle, likewise with sides forming a PPT. If we start with the smallest possible PPT, viz. (3,4,5), then we can draw three branches leading to the three PPT triangles

constructed above, and then, from each of these, we can construct three more branches by the same method, and so on.

Figure 3 shows the first few branches and vertices in the resulting “tree”. The three branches rising from each vertex are always drawn so that the uppermost one comes from the excircle on the shorter leg, a, and the lowest one comes from the excircle on the hypotenuse, c.

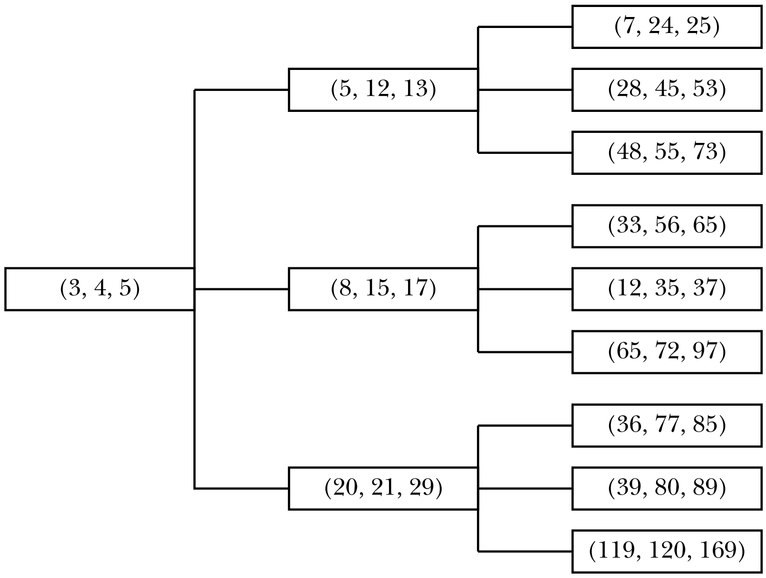

Figure 3. A tree of Primitive Pythagorean Triples.

The diagram has been called a tree and to use this in its precise mathematical sense we need to show that it contains no closed loops. If such a loop exists, then it has a unique PPT (x, y, z) say, lying on the highest level in the diagram containing PPTs in the loop. If the lowest level of the diagram containing a PPT in the loop lies above the base PPT (3, 4, 5) then extend the paths in the loop leading to (x, y, z) downwards to (3, 4, 5). This will produce two or more construction sequences leading from (3, 4, 5) to (x, y, z).

We now specify an algorithm for back tracking from (x, y, z) which at each step, yields a unique PPT at the next lowest level. This shows that loops are not possible.

Starting Point: (x, y, z) is a PPT. If $(x, y, z) = (3, 4, 5)$ we stop and there is nothing to prove.

Compute the diameters of the incircle of the PPT (x, y, z): $s = x + y - z$.

If $2s > y$ then (x, y, z) came from the excircle on the hypotenuse of the triangle (a, b, c) where $(a, b, c) = (2x + y - 2z,\ x + 2y - 2z,\ -2x - 2y + 3z)$ and is one level below (x, y, z). Return to starting point.

Else if $2s < y$ then (x, y, z) came from an excircle on one of the legs of a triangle (a, b, c) one level below (x, y, z). The correct leg is determined as follows:

If $4s < x + y$, then (x, y, z) came via the excircle construction on the *shorter* leg a of the triangle and $(a, b, c) = (x + 2y - 2z, -2x - y + 2z, -2x - 2y + 3z)$.
Return

Else $4s > x + y$, and then (x, y, z) came via the excircle construction on the *longer* leg b of the triangle and $(a, b, c) = (-2x - y + 2z, x + 2y + 2z, -2x - 2y + 3z)$
Return

It is readily verified that if (x, y, z) is a PPT for which $2s = y$ then $(x, y, z) = (3, 4, 5)$ and that there is no PPT for which $4s = x + y$, meaning that the above algorithm applies non trivially to any PPT $(x, y, z) \neq (3, 4, 5)$.

It is also readily verified that in all three cases, the algorithm ensures that $a < b < c$, and in particular distinguishes between excircle constructions on the longer and shorter legs.

Finally the triangle inequality $x + y > z$ immediately shows that $z > c$ and this implies that, after a finite number of applications, the algorithm gives a unique "reverse" sequence from (x, y, z) to $(3, 4, 5)$.

Conclusion

What is particularly appealing about this example is that it is, we contend, truly in the spirit of Fermat, in that we create a specific "infinite ascent" from a single PPT base, and then construct a "finite descent" down to that base in order to show we have captured everything!

References

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